An Introduction to Stochastic Partial Differential Equations

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29. August 2014
Outline

- SPDE
- Existence and Uniqueness of Solution
- A Stochastic Heat Equation
SPDE is an interdisciplinary area at the crossroads of stochastic processes and partial differential equations.

Wave equation

\[
\frac{\partial^2 u(t, x)}{\partial t^2} = \kappa \frac{\partial^2 u(t, x)}{\partial x^2} + F(t, x), \quad t \geq 0, \quad 0 \leq x \leq L
\]  

(1)

If \( F \) is a random noise, (1) can be interpreted as a guitar in the desert.

For example, \( F(t, x) = W(t, x) \) is the space-time white noise. Heuristically, \( W(t, x) \) is a (Gaussian) random field such that

\[
\mathbb{E}(W(t, x)W(s, y)) = \delta(t - s)\delta(x - y).
\]

In this case, (1) does not have classical meaning and must be interpreted as an infinite dimensional integral equation.
Our topic: Semilinear parabolic problems driven by additive Brownian noise.

Our approach: Hilbert space approach based on the theory of operator semigroup. See [DaPrato-Zabczyk] and [Prevot-Röckner].

This is an (elementary) introduction since we do not consider

1. Equations with multiplicative noise
2. Equations driven by fractional Brownian noise (Gaussian, non-Markov, non-semimartingale)
3. Equations driven by non-Gaussian noise (e.g. Levy noise, alpha-stable noise)
4. Equations with rough (non-Lipschitz) nonlinearities
5. Variational solution in Gelfand triplets
6. Hyperbolic and Elliptic problems
7. Malliavian calculus approach and densities of solution
8. Hida calculus approach (Wick type equations)
9. Solutions via Dirichlet Forms
10. Numerical methods
11. etc
Our approach: An SPDE is translated into a stochastic evolution equation (Cauchy problem) in some infinite dimensional Banach space.

There are some crucial problems due to $\infty$-dimension!

*Infinite dimensional Lebesgue measure does not exist!*

Escape from problem: Gaussian measure!

**Theorem (Minlos-Sazanov)**

Let $H$ be a separable Hilbert space. Let $Q$ be a positive definite, symmetric, trace-class operator in $H$ and let $m \in H$. Then there exists a Gaussian measure $\mu = \mathcal{N}(m, Q)$ on $(H, \mathcal{B}(H))$ given via

$$
\hat{\mu}(h) := \int_H e^{i \langle h, u \rangle} \mu(du) = e^{i \langle m, h \rangle - \frac{1}{2} \langle Q h, h \rangle}, \quad h \in H.
$$

**Importance:** to define Hilbert-space valued Brownian motion $B$ and, hence, to construct Hilbert-space valued Itô integral with respect to $B$.
Existence result is a priori not clear!

**Theorem (Peano)**

For each continuous function $f : \mathbb{R} \times B \to B$ defined on some open set $V \subset \mathbb{R} \times B$ and for each point $(t_0, x_0) \in V$ the Cauchy problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution which is defined on some neighborhood of $t_0$.

**Theorem (Godunov)**

Each Banach space in which Peano’s theorem is true is finite dimensional.
The Stochastic Evolution Equation

Setting:
- $H$ and $U$ are two separable Hilbert spaces
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space
- $B : [0, T] \times \Omega \to U$ is a trace-class Wiener process on $U$ adapted to a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$
- $A : \text{dom}(A) \subset H \to H$ is a densely defined, self-adjoint and positive definite linear operator with compact inverse.

Aim:
existence and uniqueness of a predictable stochastic process $X : [0, T] \times \Omega \to H$
which solves the semilinear stochastic evolution equation driven by the Wiener process $B$

$$dX(t) + (AX(t) + f(t, X(t))) \ dt = g(t, X(t)) \ dB(t), \quad 0 \leq t \leq T$$
$$X(0) = X_0,$$

for some nice functions $f$ and $g$. 
Existence and Uniqueness of Solution
Stochastic evolution equation (SEE):

\[
dX(t) + (AX(t) + f(t, X(t))) \ dt = g(t, X(t)) \ dB(t), \quad 0 \leq t \leq T
\]
\[
X(0) = X_0,
\]

Definition (of mild solution of SEE)

Let \( p \geq 2 \). A predictable stochastic process \( X : [0, T] \times \Omega \rightarrow H \) is called a \( p \)-fold integrable mild solution of SEE if

\[
\sup_{t \in [0, T]} \| X(t) \|_{L^p(\Omega; H)} < \infty
\]

and, for all \( t \in [0, T] \), it holds \( \mathbb{P} \)-a.s.

\[
X(t) = E(t)X_0 - \int_0^t E(t-s)f(s, X(s)) \, ds + \int_0^t E(t-s)g(s, X(s)) \, dB(s),
\]

where \( (E(t))_{t \in [0, \infty)} \) is the analytic semigroup on \( H \) generated by \(-A\), the first integral is a Bochner integral and the second integral is the Hilbert-space valued \( \text{Itô} \) integral.
Theorem (DaPrato-Zabczyk)

Under some measurability, $L^p$-regularity and linear growth conditions on $X_0$, $f$ and $g$, there exists a unique $p$-fold integrable mild solution $X : [0, T] \times \Omega \to H$ to SEE such that for every $t \in [0, T]$ and every $s \in [0, 1)$ it holds that

$$P \left( X(t) \in H^s \right) = 1 \text{ with }$$

$$\sup_{t \in [0, T]} \| X(t) \|_{L^p(\Omega; H^s)} < \infty,$$

where $H^s$ is the Hilbert space given by $\text{dom}(A^{s/2})$. Furthermore, for every $\delta \in (0, \frac{1}{2})$ there exists a constant $CY > 0$ with

$$\| X(t_1) - X(t_2) \|_{L^p(\Omega; H)} \leq C |t_1 - t_2|^{\delta}$$

for all $t_1, t_2 \in [0, T]$.

Uniqueness here is in the sense of modification of stochastic processes.
A Stochastic Heat Equation
Stochastic heat equation with additive noise on the unit interval

**Setting:**
- $H = L^2([0, 1], \mathcal{B}([0, 1]), dx; \mathbb{R})$
- $B$ is a trace-class Wiener process on $H$.

**Problem:**
Find a measurable mapping $X : [0, T] \times \Omega \to \mathbb{R}$ such that

$$dX(t, x) = \frac{\partial^2}{\partial x^2} X(t, x) \, dt + dB(t, x) \quad \text{for all } t \in (0, T], x \in [0, 1]$$

$$X(t, 0) = X(t, 1) = 0 \quad \text{for all } t \in (0, T]$$

$$X(0, x) = X_0(x) \quad \text{for all } x \in [0, 1],$$

where $X_0 : \Omega \times [0, 1] \to \mathbb{R}$ is such that for almost all $\omega \in \Omega$, $X_0(\omega, \cdot)$ is a sufficiently smooth function which also satisfies the boundary conditions.
We translate into an abstract stochastic evolution equation on the Hilbert space $H$:

$$dX(t) + AX(t)\,dt = dB(t), \quad \text{for all } t \in [0, T],$$

$$X(0) = X_0,$$

where

$$A := -\frac{\partial^2}{\partial x^2}$$

with $\text{dom}(A) = H^1_0(0, 1) \cap H^2(0, 1)$.

The mild solution is then given by the stochastic process $X : [0, T] \times \Omega \to H$ with

$$X(t) = E(t)X_0 + \int_0^t E(t - s)\,dB(s)$$
References


Thank You!!